MTH 166

Lecture-26

Gradient of a Scalar Field

Topic:

Vector Differential Calculus

Learning Outcomes:

1. To find gradient of a scalar field

2. To find normal and unit normal vector to a surface

3. To find angle between two surfaces

<u>Gradient of a scalar field</u>

Let us consider the following vector operator called as **Del** defined as:

$$\vec{\nabla} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Let $f(x, y, z)$ be a scalar field.(Not a vector quantity)

Then, gradient of a scalar field $f(x, y, z)$ is defined as:

$$grad(f) = \vec{\nabla} f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f$$

$$= \hat{\imath}\frac{\partial f}{\partial x} + \hat{\jmath}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}$$

Compute the Gradient of the scalar function and evaluate it at the indicated point.

Problem 1. $x^3 - 3x^2y^2 + y^3$ at $(1,2)$

Solution. Let $f = x^3 - 3x^2y^2 + y^3$

$\Rightarrow f_x = 3x^2 - 6xy^2$ and $f_y = -6x^2y + 3y^2$

Gradient of f is given by:

$$grad(f) = \vec{\nabla} f = \left(f_x\hat{\imath} + f_y\hat{\jmath}\right)$$

$$= (3x^2 - 6xy^2)\hat{\imath} + (-6x^2y + 3y^2)\hat{\jmath}$$

At $(\mathbf{1},\mathbf{2})$: $\vec{\nabla} f = (3(1)^2 - 6(1)(2)^2)\hat{\imath} + (-6(1)^2(2) + 3(2)^2)\hat{\jmath}$

$= -24\hat{\imath}$ **Answer.**

Problem 2. $\log(x+y+z)$ at $(1,2.-1)$

Solution. Let $f = \log(x+y+z)$

$$\Rightarrow f_x = f_y = f_z = \frac{1}{(x+y+z)}$$

Gradient of f is given by:

$$grad(f) = \vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}\right)$$

$$= \frac{1}{(x+y+z)}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$$

At $(\mathbf{1}, \mathbf{2}, -\mathbf{1})$: $\vec{\nabla} f = \frac{1}{(1+2-1)}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$

$$= \frac{1}{2}\left(\hat{\imath} + \hat{\jmath} + \hat{k}\right)$$ **Answer.**

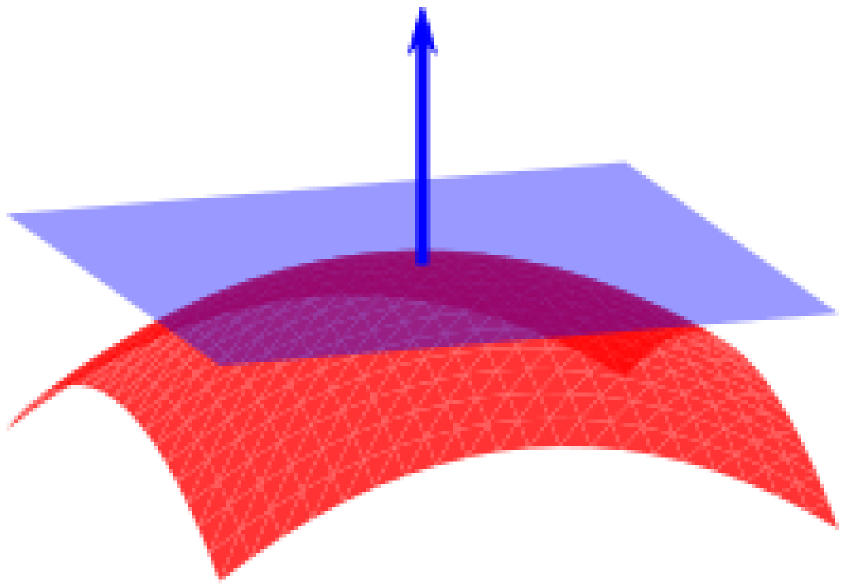
Geometrical interpretation of Gradient

Geometrically, Gradient of a scalar field represents vector normal to the surface.

Normal Vector $\vec{n} = grad(f) = \vec{\nabla} f$

Unit normal $\hat{n} = \frac{\vec{n}}{|\vec{n}|}$

$$\hat{n} = \frac{grad(f)}{|grad(f)|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$$



Find the normal vector and unit normal vector to the given curve/surface at the indicated point:

Problem 1. $y^2 = 16x$ at $(4,8)$

Solution. Let $f = 16x - y^2$

$\Rightarrow f_x = 16$ and $f_y = -2y$

Gradient of f is given by:

$$grad(f) = \vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath}\right)$$

$$= (16)\hat{\imath} + (-2y)\hat{\jmath}$$

At $(\mathbf{4}, \mathbf{8})$**:** $\vec{\nabla} f = (16\ )\hat{\imath} + (-\ 2(8))\hat{\jmath}$

$$= 16\hat{\imath} - 16\hat{\jmath}$$

Normal Vector $\vec{n} = \vec{\nabla} f = 16\hat{\imath} - 16\hat{\jmath}$

Unit normal vector $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|}$

$$\Rightarrow \hat{n} = \frac{(16\hat{\imath} - 16\hat{\jmath})}{\sqrt{(16)^2 + (-16)^2}}$$

$$\Rightarrow \hat{n} = \frac{16(\hat{\imath} - \hat{\jmath})}{16\sqrt{1+1}} = \frac{(\hat{\imath} - \hat{\jmath})}{\sqrt{2}}$$

Problem 2. $x^2 + y^2 + z^2 = 4$ at (1,1,1)

Solution. Let $f = x^2 + y^2 + z^2 - 4$

$\Rightarrow f_x = 2x$ and $f_y = 2y$ and $f_z = 2z$

Gradient of f is given by:

$$grad(f) = \vec{\nabla} f = \left(f_x \hat{\imath} + f_y \hat{\jmath} + f_z \hat{k}\right)$$

$$= (2x)\hat{\imath} + (2y)\hat{\jmath} + (2z)\hat{k}$$

At $(\mathbf{1},\mathbf{1},\mathbf{1})$: $\vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Normal Vector $\vec{n} = \vec{\nabla} f = 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Unit normal vector $\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{2\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)}{2\sqrt{3}}$

$= \frac{\left(\hat{\imath}+\hat{\jmath}+\hat{k}\right)}{\sqrt{3}}$ **Answer.**

Angle between two Surfaces

Angle between two surfaces f and g is same as the angle between their normal.

Let $\overrightarrow{n_1}$ be normal to surface f and $\overrightarrow{n_2}$ be normal to surface g.

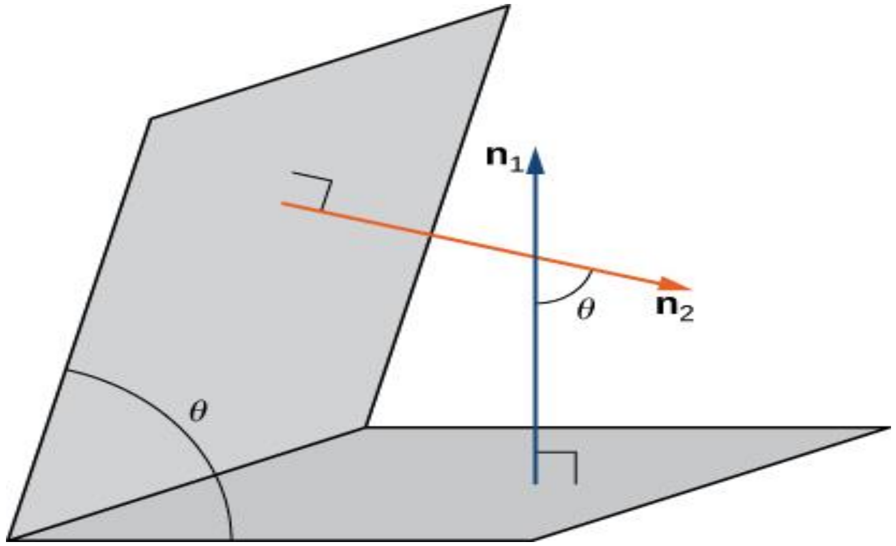
$\overrightarrow{n_1} = \vec{\nabla} f$ and $\overrightarrow{n_2} = \vec{\nabla} g$

Then. Angle between surfaces f and g is given by:

$\cos\theta = [\widehat{n_1}.\widehat{n_2}]$

$\Rightarrow \cos\theta = \left[\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}.\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}\right]$

$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla} f}{|\vec{\nabla} f|}.\frac{\vec{\nabla} g}{|\vec{\nabla} g|}\right]$



Problem. Find the angle between two given surfaces at the indicated point of intersection: $z = x^2 + y^2,\ z = 2x^2 - 3y^2$ at $(2,1,5)$.

Solution. Let $f = x^2 + y^2 - z$

$\Rightarrow f_x = 2x$ and $f_y = 2y$ and $f_z = -1$

Gradient of f is given by:

$$grad(f) = \vec{\nabla} f = \left(f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}\right)$$

$$= (2x)\hat{\imath} + (2y)\hat{\jmath} + (-1)\hat{k}$$

At $(\mathbf{2}, \mathbf{1}, \mathbf{5})$: $\vec{\nabla} f = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$

Normal Vector $\overrightarrow{n_1} = \vec{\nabla} f = 4\hat{\imath} + 2\hat{\jmath} - \hat{k}$

$z = 2x^2 - 3y^2$ at $(2,1,5)$.

Again, Let $g = 2x^2 - 3y^2 - z$

$\Rightarrow g_x = 4x$ and $g_y = -6y$ and $g_z = -1$

Gradient of g is given by:

$$grad(g) = \vec{\nabla} g = \left(g_x \hat{\imath} + g_y \hat{\jmath} + g_z \hat{k}\right)$$

$$= (4x)\hat{\imath} + (-6y)\hat{\jmath} + (-1)\hat{k}$$

At $(\mathbf{2}, \mathbf{1}, \mathbf{5})$: $\vec{\nabla} g = 8\hat{\imath} - 6\hat{\jmath} - \hat{k}$

Normal Vector $\overrightarrow{n_2} = \vec{\nabla} g = 8\hat{\imath} - 6\hat{\jmath} - \hat{k}$

We know, the angle between two surfaces f and g is same as the angle between their normal.

Then. Angle between surfaces f and g is given by:

$$\cos\theta = [\widehat{n_1}.\widehat{n_2}] \qquad \Rightarrow \cos\theta = \left[\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}.\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla}f}{|\vec{\nabla}f|}.\frac{\vec{\nabla}g}{|\vec{\nabla}g|}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{(4\hat{\imath}+2\hat{\jmath}-\hat{k})}{\sqrt{21}}.\frac{(8\hat{\imath}-6\hat{\jmath}-\hat{k})}{\sqrt{101}}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{(32-12+1)}{\sqrt{21}\sqrt{101}}\right] = cos^{-1}\left[\frac{21}{\sqrt{21}\sqrt{101}}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\sqrt{\frac{21}{101}}\right]$$ **Answer.**